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On Polynomial "Interpolation" in L_1

BORIS SHEKHTMAN

Department of Mathematics, University of South Florida, Tampa, Florida, U.S.A.

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We study operators F from $L_1[-\pi, \pi]$ into the space of trigonometric polynomials of degree $m \geqslant n$ that satisfy n additional conditions

$$g_k(Ff) = g_k(f),$$

where $g_k \in L_1[-\pi, \pi]$; supp $g_k \cap$ supp $g_j = \varnothing$ for $k \neq j$. It is well known that for $m = n$

$$\|F\|_{L_1 \to L_1} \geqslant C \cdot \log n.$$

We show that for arbitrary $m > n$

$$\|F\|_{L_1 \to L_1} \geqslant C \cdot \log \frac{n}{m-n},$$

where C is independent of $g_1, ..., g_n$. © 1991 Academic Press, Inc.

1. STATEMENT OF THE PROBLEM

This paper is a by-product of an effort to solve a problem posed by J. Szabados in [3]. Let $\mathscr{T}_m := \text{span}\{1, e^{it}, ..., e^{imt}\}$ denote the space of trigonometric polynomials on $[-\pi, \pi]$ of degree m. Let $t_1, ..., t_n$ $(n < m)$ be a set of points in $(-\pi, \pi)$. We use δ_{t_i} to denote the linear functional of point evaluation at t_i.

We consider the set $\mathscr{F}_n$ of operators (not necessarily linear)

$$F: C_{[-\pi, \pi]} \to \mathscr{T}_m$$

which satisfy $\delta_{t_i}(Ff) = \delta_{t_i}(f)$ for $i = 1, ..., n$.

It was conjectured in [3] that there exists a universal constant $C > 0$ such that

$$\|F\|_{L_\infty \to L_\infty} \geqslant C \cdot \log \left(\frac{n}{m-n}\right) \tag{1.1}$$

for all $F \in \mathscr{F}_n$ and all choices of points $t_1, ..., t_n$.

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In [2] the author proved (1.1) for all linear operators in $\mathscr{F}_n$. The non-linear situation remains open. Note that constraints on the set $\mathscr{F}_n$ are posed by the functional $\delta_{t_1}, ..., \delta_{t_r}$ which satisfy $\operatorname{supp} \delta_{t_k} \cap \operatorname{supp} \delta_{t_j} = \varnothing$, for all $k \neq j$.

In this paper we consider an analogous problem for "interpolation" in $L_1[-\pi, \pi]$.

For an arbitrary operator $F: L_1 \to L_1$ define $\|F\| := \sup\{\|Ff\|_{L_1} : \|f\|_{L_1} \leqslant 1\}$. The symbol $\mathscr{T}_m$ again stands for trigonometric polynomials of degree m. Let $g_1, ..., g_n$ be any n nontrivial functions in $L_\infty[-\pi, \pi]$ such that $\operatorname{supp} g_k \cap \operatorname{supp} g_j = \varnothing$ for $j \neq k$.

Let $\mathscr{F}_n$ be the class of operators

$$F: L_1[-\pi, \pi] \to \mathscr{T}_m$$

satisfying

$$\int_{-\pi}^{\pi} g_k(t)(Ff)(t)\, dt = \int_{-\pi}^{\pi} g_k(t)\, f(t)\, dt$$

for all $f \in L_1[-\pi, \pi]$ and all $k = 1, ..., n$.

THEOREM. *There exists a universal constant* $C > 0$ *such that*

$$\|F\|_{L_1 \to L_1} \geqslant C \cdot \log\left(\frac{n}{m-n}\right) \qquad \forall F \in \mathscr{F}_n$$

(*C does not depend on the choice of* $g_1, ..., g_n$).

The proof of this result is based on the observation that in $L_1[-\pi, \pi]$ the best choice for F is a linear operator. The inequality (1.2) for linear operators is established the same way as in [2].

2. Proofs

In what follows we assume (without loss of generality) that $\|g_k\|_{L_\infty} = 1$ and $m = n + q$ where q is an integer, $q < n$. We will use two lemmas.

The first one is a simple consequence of the Hardy inequality.

LEMMA 1 (cf. [1, 2]). *Let F be a* linear *operator from* $L_1[-\pi, \pi]$ *into* $\mathscr{T}_m$ *given by*

$$(Ff)(\theta) = \sum_{j=0}^{m} \left(\int u_j(t)\, f(t)\, dt\right) e^{ij\theta}; \qquad u_j \in L_\infty[-\pi, \pi].$$

Then

$$\|F\|_{L_1 \to L_1} \geqslant \frac{1}{\pi} \sum_{j=0}^{m} \frac{\|u_j\|_{L_1}}{j+1}.$$

LEMMA 2 (cf. [2]). *Let A and U be $n \times (n+q)$ and $(n+q) \times n$ matrices*

$$A = (a_{ij})_{i=1, j=1}^{n, n+q}; \qquad U = (u_{ij})_{i=1, j=1}^{n+q, n}.$$

Suppose that $|a_{kj}| \leqslant 1$ $(k = 1, ..., n;\ j = 1, ..., n+q)$ and $A \cdot U = I$, the $n \times n$ identity matrix.

Then there are $n - q$ rows of matrix U indexed by $k_1, ..., k_{n-q}$ such that

$$\sum_{j=1}^{n} |u_{k_l j}| \geqslant \frac{1}{2}; \qquad l = 1, ..., n-q. \tag{2.1}$$

Remark. The meaning of Lemma 2 becomes clear by considering the case where $q = 0$. Then A and U are square matrices with $AU = I$. Hence $UA = I$ and writing out the diagonal of UA we have

$$1 = \sum_{j=1}^{n} a_{jk} u_{kj} \leqslant \left(\sum_{j=1}^{n} |u_{kj}| \right) \max |a_{jk}| \leqslant \sum_{j=1}^{n} |u_{kj}|,$$

for all $k = 1, ..., n$. Thus there are $n - q = n$ rows in U with the property (2.1).

Proof of the Theorem. Let γ_n denote the quantity

$$\gamma_n = \inf\{ \|F\|_{L_1 \to L_1} : F \in \mathscr{F}_n \}.$$

Pick $\varepsilon > 0$ such that $0 < \varepsilon < 1/(\gamma_n + 2)$. Let $F \in \mathscr{F}_n$ be a fixed operator with $\|F\| - \gamma_n < \varepsilon^2$. Since $[L_1(\text{supp } g_k)]^* = L_\infty(\text{supp } g_k)$ there exist functions $f_k \in L_1(\text{supp } g_k)$ such that $\int f_k g_k = 1$; $\|f_k\|_{L_1} < 1 + \varepsilon^2$. Let $p_n = F f_k$. Consider the *linear* operator from $L_1[-\pi, \pi]$ into $\mathscr{T}_m$ defined by

$$Pf = \sum_{k=1}^{n} \left(\int f \cdot g_k \right) p_k.$$

Clearly $\int g_k p_j = \int g_k F f_j = \int g_k f_j = \delta_{jk}$. Hence P is a linear projection and $P \in \mathscr{F}_n$.

Also

$$\|p_k\|_{L_1} = \|F f_k\| \leqslant (\gamma_n + \varepsilon^2)(1 + \varepsilon^2) < \gamma_n + \varepsilon.$$

Hence

$$\|P\|_{L_1 \to L_1} = \sup_t \int \left| \sum_{k=1}^{n} g_k(t)\, p_k(s) \right| ds = \max_k \int |g_k(t)|\, |p_k(s)|\, ds$$
$$\leqslant \gamma_n + \varepsilon.$$

We now estimate $\|P\|_{L_1 \to L_1}$ from below. Since P is a linear operator from L_1 into $\mathscr{T}_m$ it can be written as

$$(Pf)(\theta) = \sum_{k=1}^{m} \left(\int f \cdot u_k \right) e^{ik\theta}.$$

Since P is a projection, so is P^*. And since $\int (Pf)\, g_k = \int f g_k$ we have $g_k \in \text{Range } P^*$. Hence

$$\sum_{k=1}^{m} u_k(s) \left(\int e^{ik\theta} \tilde{g}_j(\theta)\, d\theta \right) = \tilde{g}_j(s),$$

where $\tilde{g}_i = g_j / \|g_i\|_{L_1}$.

Denote

$$a_{jk} = \int e^{ik\theta} \tilde{g}_j(\theta)\, d\theta.$$

Then

$$|a_{jk}| \leqslant \|\tilde{g}_j\|_{L_1} \leqslant 1.$$

We have

$$\sum a_{jk} u_k(s) = \tilde{g}_j(s).$$

Let $\bar{f}_l \in L_\infty(\text{supp } g_l)$ such that $\|\bar{f}_l\|_{L_\infty} \leqslant 1$ and

$$\int \bar{f}_l(s)\, g_k(s)\, ds = \delta_{lk}.$$

Then

$$\sum_{k=1}^{m} a_{jk} \int u_k \bar{f}_l = \delta_{jl} \qquad \text{for all } j, \qquad l = 1, \ldots, n.$$

Letting

$$u_{kl} := \int u_k \bar{f}_l$$

we have

$$\sum_{k=1}^{m} a_{jk} u_{kl} = \delta_{jl} \qquad \text{for all} \quad j, l = 1, ..., n.$$

By Lemma 2 there exist $k_1, ..., k_{n-q}$ integers between 1 and $n+q$ such that

$$\sum_{j=1}^{n} |u_{k_r, j}| \geqslant \frac{1}{2}.$$

Hence

$$\frac{1}{2} \leqslant \sum_{j=1}^{n} |u_{k_r, j}| = \sum_{j=1}^{n} \left| \int u_{k_r}(s) \bar{f}_j(s)\, ds \right| \leqslant \int \left| \sum_{j=1}^{n} \bar{f}_j(s) \right| |u_{k_r}(s)|\, ds$$

$$\leqslant \|u_{k_r}\|_{L_1} \cdot \left\| \sum \bar{f}_j(s) \right\|_{L_\infty} = \|u_{k_r}\|_{L_1}.$$

In other words, among the functions $u_1, ..., u_m$ there are $n-q$ functions $u_{k_1}, ..., u_{k_{n-q}}$ with L_1 norm greater than or equal to $\frac{1}{2}$.

By Lemma 1 and the previous remark

$$\gamma_n + \varepsilon \geqslant \|P_n\|_{L_1 \to L_1} \geqslant \frac{1}{\pi} \sum_{j=1}^{n+q(n)} \frac{\|u_j\|_{L_1}}{j} \geqslant \frac{1}{2\pi} \sum_{j=2q(n)}^{n+q(n)} \frac{1}{j}$$

$$\geqslant C \cdot \log \frac{n+q(n)}{2q(n)}$$

which gives us the conclusion of the theorem. ∎

Remark. The same theorem holds if we replace the complex polynomial by real trigonometric polynomials. In the real trigonometric case we need to use an operator analog of the Sidon inequality instead of Lemma 1 (cf. 1, 2 for details).

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